DOI:
10.1049/iet-cta.2015.0513

Document Version
Peer reviewed version

Link to publication record in King's Research Portal

Citation for published version (APA):
Cao, K., Gao, X., Lam, H-K., & Vasilakos, A. (2016). H fuzzy PID control synthesis for Takagi-Sugeno fuzzy systems. *Iet Control Theory And Applications*, *10*(6), 607-616. https://doi.org/10.1049/iet-cta.2015.0513

H_∞ Fuzzy PID Control Synthesis for Takagi-Sugeno Fuzzy Systems

Kairui Cao [1*], X. Z. Gao [2], H. K. Lam [3], A. V. Vasilakos [4]

[1] National Key Laboratory of Tunable Laser Technology, Harbin Institute of Technology, Harbin, P. R. China
[2] Department of Electrical Engineering and Automation, Aalto University School of Electrical Engineering, Aalto, Finland
[3] Department of Informatics, King's College London, London, United Kingdom
[4] Computer Science Department, Kuwait University, Kuwait City, Kuwait
[*] Email: caokairui1984@139.com

Abstract: This paper proposes a kind of H_∞ fuzzy PID control synthesis method for Takagi-Sugeno (T-S) fuzzy systems. The basic idea of the presented method is to transform the fuzzy PID controller design problem into that of the fuzzy static output feedback (SOF) controller design. Based on an iterative linear matrix inequality (ILMI) algorithm, the fuzzy SOF control laws can be obtained. After that, the fuzzy PID controller is recovered from the fuzzy SOF controller. Simulation examples are given to show the effectiveness of the proposed method.

1. Introduction

Although many sophisticated control theories and techniques have been developed in the past decades, PID controllers are still extensively used in more than 90% of industrial processes [1]. A PID controller can often provide remarkable control performances for linear systems. The structure of a PID controller is rather simple: the input of the controller is the control error, and the output is the sum of three terms, i.e., the proportional term (proportional to the error), the integral term (proportional to the integral of the error), and the derivative term (proportional to the derivative of the error). The gains of the above three terms need to be determined such that the closed-loop system is stable with prescribed performance. Many different methods have been proposed for tuning PID controllers, such as the Ziegler-Nichols method [2] and the optimum tuning method [3].

In order to achieve better performances and meanwhile preserve the simple structure of PID controllers, conventional PID controllers have been extended to fuzzy PID controllers [4]. There are two main types of fuzzy controllers, i.e., Mamdani type and Takagi-Sugeno (T-S) type. The difference between them is the form of fuzzy logic rule consequent parts. A Mamdani fuzzy controller uses fuzzy sets as the rule consequent, whereas a T-S fuzzy controller uses the linear functions of input variables. Significant efforts have been made to investigate the Mamdani fuzzy PID control systems. To mention a few, the analytical structure analysis results are reported in [5-7], the stability analysis problems are considered in [8, 9], and the controller design methods are proposed in [10-15]. For the T-S fuzzy PID control systems, the analytical structure analysis is explored in [16, 17], where the results show that a T-S fuzzy PID controller is a nonlinear PID controller with variable gains. The stability analysis of T-S fuzzy PID control systems is investigated based on the small gain theorem [17], the passivity theory [18], the describing function method [19], and the circle criterion [20, 21], respectively. The T-S fuzzy PID controllers are applied to human arterial pressure control [16] and motor control [22]. It is worth noting that the controlled plants in [16, 18-22] are linear systems rather than nonlinear systems. In [23], a neural network-based learning method is used for tuning the parameters of the fuzzy PID controllers. However, the stability analysis is not considered. To the best of the authors' knowledge, there little work discussing how to design a fuzzy PID controller for nonlinear systems with guaranteed asymptotical stability and H_∞ performance. In [24], a fuzzy PI controller design method for T-S fuzzy systems is developed via an iterative linear matrix inequality (ILMI) algorithm. Although the stability of the closed-loop system can be guaranteed, the method is based on the assumption that some system matrices of the T-S fuzzy model are common (i.e. input matrices $\boldsymbol{B}_{21} = \boldsymbol{B}_{22} = \cdots = \boldsymbol{B}_{2r}$ and output matrices $\boldsymbol{C}_{21} = \boldsymbol{C}_{22} = \cdots = \boldsymbol{C}_{2r}$), which actually limits the applicability of this method.

In this paper, the focus is on the H_∞ fuzzy PID control synthesis for T-S fuzzy systems. Our main contribution is that a systematic way of designing fuzzy PID controllers is proposed such that the closed-loop system is stable with a prescribed H_∞ performance. The basic idea of the developed method is to transform the problem of fuzzy PID controller design into that of fuzzy static output feedback (SOF) controller design. For the latter problem, the well-established results can be used to derive the fuzzy SOF controller. After that, the fuzzy PID controller can be recovered from the fuzzy SOF controller. In fact, similar ideas have been extensively used in the design of linear PID controllers, such as [25, 26], where it is noted that the inverse of a combination of system matrices needs to be used. Since the system matrices of a T-S fuzzy model are usually not constant, these methods cannot be generalized to the fuzzy PID controller design. In our approach, we introduce three kinds of state space realizations for fuzzy PID, PI, and PD controllers. Based on these state space realizations and the augmentation technique [27, 28], the H_∞ fuzzy PID, PI, and PD controller design problems can be investigated in a unified way. Three numerical simulation examples are illustrated to demonstrate the efficiency of the proposed method

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ matrices. $\boldsymbol{0}$ and $\boldsymbol{I}$ are the zero matrix and identity matrix with appropriate dimensions, respectively. For a matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, the notation $\boldsymbol{P}^T$ represents the transpose of $\boldsymbol{P}$, and $\boldsymbol{P} > 0$ means that $\boldsymbol{P}$ is a real symmetric positive definite matrix. L_2 is the space which consists of all Lebesgue measurable functions $\boldsymbol{f}(t) \in \mathbb{R}^p$ defined on the interval $[0, \infty)$ such that $\int_0^\infty \boldsymbol{f}^T(t)\boldsymbol{f}(t)dt < \infty$.

2. Preliminaries

The T-S fuzzy model and the fuzzy PID controller are briefly introduced in this section. The fuzzy model is used to represent the nonlinear plant to facilitate the stability analysis and controller design. A fuzzy PID controller is employed to close the feedback loop. Assume that the nonlinear plant can be described by

$$\begin{cases} \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{w} + \boldsymbol{h}(\boldsymbol{x})\boldsymbol{u} \\ \boldsymbol{z} = \boldsymbol{\alpha}(\boldsymbol{x}) + \boldsymbol{\xi}(\boldsymbol{x})\boldsymbol{w} \\ \boldsymbol{y} = \boldsymbol{\beta}(\boldsymbol{x}) \end{cases} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{z} \in \mathbb{R}^{n_z}$, and $\boldsymbol{y} \in \mathbb{R}^{n_y}$ are the state vector, the controlled output, and the measured output, respectively; $\boldsymbol{w} \in \mathbb{R}^{n_w}$ is the disturbance that belongs to the L_2 space and $\boldsymbol{u} \in \mathbb{R}^{n_u}$ is the control input; $\boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^n$, $\boldsymbol{g}(\boldsymbol{x}) \in \mathbb{R}^{n \times n_w}$, $\boldsymbol{h}(\boldsymbol{x}) \in \mathbb{R}^{n \times n_u}$, $\boldsymbol{\alpha}(\boldsymbol{x}) \in \mathbb{R}^{n_z}$, $\boldsymbol{\xi}(\boldsymbol{x}) \in \mathbb{R}^{n_z \times n_w}$, and $\boldsymbol{\beta}(\boldsymbol{x}) \in \mathbb{R}^{n_y}$ are functions of $\boldsymbol{x}$.

2.1. T-S fuzzy model

The T-S fuzzy model is described by IF-THEN rules, which represent local linear input-output relations of nonlinear plant (1) [29]. Assume that the T-S fuzzy model consists of r fuzzy logic rules, and the ith rule is of the following form:

Rule i: IF $f_1(\boldsymbol{x})$ is M_1^i AND … AND $f_\psi(\boldsymbol{x})$ is M_ψ^i, THEN

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{B}_{1i} & \boldsymbol{B}_{2i} \\ \boldsymbol{C}_{1i} & \boldsymbol{D}_{1i} & \boldsymbol{0} \\ \boldsymbol{C}_{2i} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \\ \boldsymbol{u} \end{bmatrix} \tag{2}$$

where M_α^i is a fuzzy set of rule i corresponding to the known function $f_\alpha(\boldsymbol{x})$, where $\alpha = 1, 2, \ldots, \psi$, ψ is an integer, and $i = 1, 2, \ldots, r$; $\boldsymbol{A}_i \in \mathbb{R}^{n \times n}$, $\boldsymbol{B}_{1i} \in \mathbb{R}^{n \times n_w}$, $\boldsymbol{B}_{2i} \in \mathbb{R}^{n \times n_u}$, $\boldsymbol{C}_{1i} \in \mathbb{R}^{n_z \times n}$, $\boldsymbol{D}_{1i} \in \mathbb{R}^{n_z \times n_w}$, and $\boldsymbol{C}_{2i} \in \mathbb{R}^{n_y \times n}$ are the system matrices. The dynamics of the T-S fuzzy model can be described by

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}(h) & \boldsymbol{B}_1(h) & \boldsymbol{B}_2(h) \\ \boldsymbol{C}_1(h) & \boldsymbol{D}_1(h) & \boldsymbol{0} \\ \boldsymbol{C}_2(h) & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \\ \boldsymbol{u} \end{bmatrix} \tag{3}$$

where $A(h) = \sum_{i=1}^{r} h_i(x)A_i$, $B_1(h) = \sum_{i=1}^{r} h_i(x)B_{1i}$, $B_2(h) = \sum_{i=1}^{r} h_i(x)B_{2i}$, $C_1(h) = \sum_{i=1}^{r} h_i(x)C_{1i}$, $D_1(h) = \sum_{i=1}^{r} h_i(x)D_{1i}$, $C_2(h) = \sum_{i=1}^{r} h_i(x)C_{2i}$, and

$$h_i(\boldsymbol{x}) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\boldsymbol{x})) \times \mu_{\mathrm{M}_2^i}(f_2(\boldsymbol{x})) \times \cdots \times \mu_{\mathrm{M}_\psi^i}(f_\psi(\boldsymbol{x}))}{\sum_{k=1}^{r}(\mu_{\mathrm{M}_1^k}(f_1(\boldsymbol{x})) \times \mu_{\mathrm{M}_2^k}(f_2(\boldsymbol{x})) \times \cdots \times \mu_{\mathrm{M}_\psi^k}(f_\psi(\boldsymbol{x})))} \geq 0 \tag{4}$$

where $\mu_{\mathrm{M}_\alpha^k}(f_\alpha(\boldsymbol{x}))$ is the grade of membership of $f_\alpha(\boldsymbol{x})$ in M_α^k. From (4), we have $\sum_{i=1}^{r} h_i(\boldsymbol{x}) = 1$.

2.2. Fuzzy PID controller

The fuzzy PID controller, whose structure is shown in Fig. 1, is to be designed based on the T-S fuzzy model (3). The fuzzy PID controller has r fuzzy rules and the jth rule is of the following form:

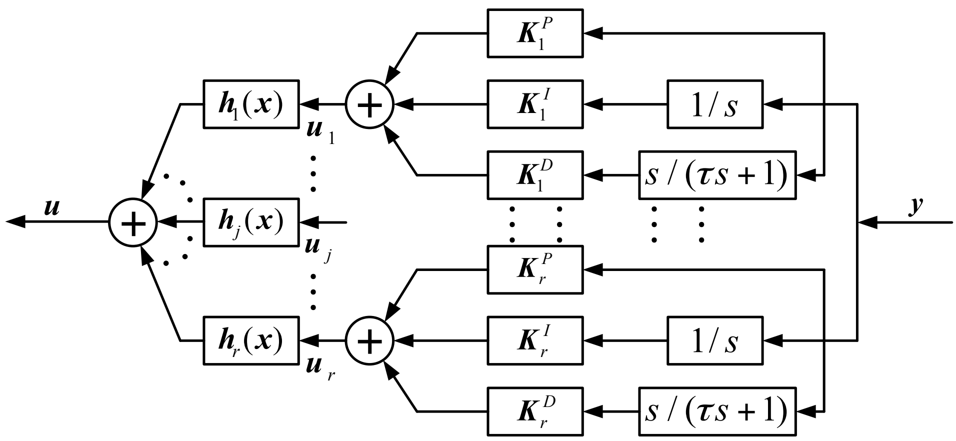


Fig. 1. *Structure of the T-S fuzzy PID controller*

Rule j: IF $f_1(\boldsymbol{x})$ is M_1^j AND … AND $f_\psi(\boldsymbol{x})$ is M_ψ^j, THEN

$$\boldsymbol{u}_j(s) = \boldsymbol{K}_j^P + \frac{\boldsymbol{K}_j^I}{s} + \frac{\boldsymbol{K}_j^D s}{\tau s + 1} \tag{5}$$

where s is the complex number frequency; τ is a known positive scalar; $\boldsymbol{K}_j^P, \boldsymbol{K}_j^I, \boldsymbol{K}_j^D \in \mathbb{R}^{n_u \times n_y}$ are coefficients of the proportional, integral, and differential terms, respectively; $\boldsymbol{u}_j(s)$ is the Laplace transform of the output $\boldsymbol{u}_j(t)$ of the local PID controller, $j = 1, 2, \cdots, r$. The output of the fuzzy controller is

$$\boldsymbol{u} = \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{u}_j(t). \tag{6}$$

It is emphasized that the fuzzy PI and PD controllers can be derived by setting $\boldsymbol{K}_j^D = 0$ and $\boldsymbol{K}_j^I = 0$, $j = 1, 2, \quad , r$, respectively.

2.3. Problem Formulation

Given a predefined H_∞ performance $\gamma > 0$, design a fuzzy PID controller, whose *j*th rule is of the form (5), such that the closed-loop is asymptotically stable and the induced L_2-norm of the channel from $\boldsymbol{w}$ to the controlled output $\boldsymbol{z}$ is less than γ^2 under zero initial conditions, i.e.,

$$\int_0^\infty \boldsymbol{z}^T \boldsymbol{z} dt / \int_0^\infty \boldsymbol{w}^T \boldsymbol{w} dt \leq \gamma^2 \tag{7}$$

for all $\boldsymbol{w} \in L_2[0, \infty)$ and $\boldsymbol{w} \neq 0$.

3. Main results

In this section, we focus on the H_∞ fuzzy PID controller design for the T-S fuzzy systems. First, we present a state space realization of the fuzzy PID controller. We next transform the fuzzy PID control system into a fuzzy SOF control system. A theorem is given to demonstrate that these two control systems are equivalent to each other in the sense of stability characteristic and H_∞ performance. An ILMI algorithm is further presented to design the fuzzy SOF controller. Finally, we recover the fuzzy PID controller from the fuzzy SOF controller.

3.1. A state space realization of fuzzy PID controller

The parameters of fuzzy PID controllers will be determined based on a time domain-based method. Thus, the fuzzy PID controller described in the frequency domain should be transformed into a state space form. To this end, we rewrite the *j*th local PID controller as follows:

$$\boldsymbol{u}_j(s) = \boldsymbol{K}_j^P + \frac{\boldsymbol{K}_j^I}{s} + \frac{\boldsymbol{K}_j^D s}{\tau s + 1} = \frac{\boldsymbol{Q}_{2j}\boldsymbol{s}^2 + \boldsymbol{Q}_{1j}s + \boldsymbol{Q}_{0j}}{\tau s^2 + s}$$

where $\boldsymbol{Q}_{2j}, \boldsymbol{Q}_{1j}, \boldsymbol{Q}_{0j} \in {}^{n_u \times n_y}$ are matrices defined by $\boldsymbol{Q}_{2j} = \boldsymbol{K}_j^D + \tau \boldsymbol{K}_j^P$, $\boldsymbol{Q}_{1j} = \boldsymbol{K}_j^P + \tau \boldsymbol{K}_j^I$, and $\boldsymbol{Q}_0 = \boldsymbol{K}^I$. Using the linear system theory [30], we can write the *j*th local PID controller by the following state space realization:

$$\begin{bmatrix} \boldsymbol{x}_c \\ \boldsymbol{u}_j \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^{PID} & \boldsymbol{B}^{PID} \\ \boldsymbol{C}_j^{PID} & \boldsymbol{D}_j^{PID} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_c \\ \boldsymbol{y} \end{bmatrix} \tag{8}$$

where $\boldsymbol{x}_c$ is the state vector of the local PID controller and

$$\boldsymbol{A}^{PID} = diag\Bigg\{\underbrace{\begin{bmatrix} -\tau^{-1} & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -\tau^{-1} & 0 \\ 1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} -\tau^{-1} & 0 \\ 1 & 0 \end{bmatrix}}_{n_y}\Bigg\}, \tag{9}$$

$$\boldsymbol{B}^{PID} = diag\Bigg\{\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{n_y}\Bigg\}, \tag{10}$$

$$\boldsymbol{C}_j^{PID} = \begin{bmatrix} \tau^{-1}\boldsymbol{Q}_{1j} - \tau^{-2}\boldsymbol{Q}_{2j} & \tau^{-1}\boldsymbol{Q}_{0j} \end{bmatrix}\boldsymbol{T},\ \boldsymbol{D}_j^{PID} = \tau^{-1}\boldsymbol{Q}_{2j}, \tag{11}$$

$$\boldsymbol{T} = \underbrace{\begin{bmatrix} [1 \quad 0] & & \\ & \ddots & \\ & & [1 \quad 0] \\ [0 \quad 1] & & \\ & \ddots & \\ & & [0 \quad 1] \end{bmatrix}}_{n_y}. \tag{12}$$

Note that $\boldsymbol{T}$ is a permutation matrix such that

$$\begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{n_y} & \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{n_y} \end{bmatrix}\boldsymbol{T} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{b}_1 & \cdots & \boldsymbol{a}_{n_y} & \boldsymbol{b}_{n_y} \end{bmatrix}$$

where $\boldsymbol{a}_i, \boldsymbol{b}_i \in \mathbb{R}^{n_y}$ are column vectors, $i = 1, 2, \cdots, n_y$. The above matrices $\boldsymbol{C}_j^{PID}$, $\boldsymbol{D}_j^{PID}$, $\boldsymbol{K}_j^P$, $\boldsymbol{K}_j^I$, and $\boldsymbol{K}_j^D$ have the following relationships:

$$\begin{cases} \boldsymbol{K}_j^P = \tau\boldsymbol{C}_{j1}^{PID} + \boldsymbol{D}_j^{PID} - \tau^2\boldsymbol{C}_{j2}^{PID} \\ \boldsymbol{K}_j^I = \tau\boldsymbol{C}_{j2}^{PID} \\ \boldsymbol{K}_j^D = \tau^3\boldsymbol{C}_{j2}^{PID} - \tau^2\boldsymbol{C}_{j1}^{PID} \end{cases} \tag{13}$$

where $\begin{bmatrix} \boldsymbol{C}_{j1}^{PID} & \boldsymbol{C}_{j2}^{PID} \end{bmatrix} = \begin{bmatrix} \tau^{-1}\boldsymbol{Q}_{1j} - \tau^{-2}\boldsymbol{Q}_{2j} & \tau^{-1}\boldsymbol{Q}_{0j} \end{bmatrix}\boldsymbol{T}$.

Remark 1: Generally, the state space realization of a linear PID controller is not unique. We choose a special form to represent the local PID controller in (8), where $\boldsymbol{A}^{PID}$ and $\boldsymbol{B}^{PID}$ are fixed constant matrices. It implies that the state vectors of all the local PID controllers are the same since $\boldsymbol{x}_c = \boldsymbol{A}^{PID}\boldsymbol{x}_c + \boldsymbol{B}^{PID}\boldsymbol{y}$ and all the local controllers have the same input $\boldsymbol{y}$. Therefore, ***the state vector $\boldsymbol{x}_c$ of the local controllers can be regarded as the state vector of the fuzzy PID controller***.

From (6) and (8), the state space realization of the fuzzy PID controller can be written as follows:

$$\begin{bmatrix} \boldsymbol{x}_c \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{x}_c \\ \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{u}_j \end{bmatrix} = \sum_{j=1}^{r} h_j(\boldsymbol{x}) \begin{bmatrix} \boldsymbol{A}^{PID} & \boldsymbol{B}^{PID} \\ \boldsymbol{C}_j^{PID} & \boldsymbol{D}_j^{PID} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_c \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^{PID} & \boldsymbol{B}^{PID} \\ \boldsymbol{C}^{PID}(h) & \boldsymbol{D}^{PID}(h) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_c \\ \boldsymbol{y} \end{bmatrix} \tag{14}$$

where $\boldsymbol{C}^{PID}(h) = \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{C}_j^{PID}$ and $\boldsymbol{D}^{PID}(h) = \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{D}_j^{PID}$.

Following the same principle of the above analysis, we can also derive the state space realization of the fuzzy PI controller as follows:

$$\begin{bmatrix} \boldsymbol{x}_c \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^{PI} & \boldsymbol{B}^{PI} \\ \boldsymbol{C}^{PI}(h) & \boldsymbol{D}^{PI}(h) \end{bmatrix} \tag{15}$$

where $\boldsymbol{A}^{PI} = \boldsymbol{0} \in \ ^{n_y \times n_y}$, $\boldsymbol{B}^{PI} = \boldsymbol{I} \in \ ^{n_y \times n_y}$, $\boldsymbol{C}^{PI}(h) = \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{C}_j^{PI}$, and $\boldsymbol{D}^{PI}(h) = \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{D}_j^{PI}$. The coefficients $\boldsymbol{K}_j^{P}$ and $\boldsymbol{K}_j^{I}$ can be obtained by

$$\begin{cases} \boldsymbol{K}_j^{P} = \boldsymbol{D}_j^{PI} \\ \boldsymbol{K}_j^{I} = \boldsymbol{C}_j^{PI}. \end{cases} \tag{16}$$

Similarly, the state space realization of the fuzzy PD controller can be written as follows:

$$\begin{bmatrix} \boldsymbol{x}_c \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^{PD} & \boldsymbol{B}^{PD} \\ \boldsymbol{C}^{PD}(h) & \boldsymbol{D}^{PD}(h) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_c \\ \boldsymbol{y} \end{bmatrix} \tag{17}$$

where $\boldsymbol{A}^{PD} = diag\{-\tau^{-1}, \quad , -\tau^{-1}\} \in \ ^{n_y \times n_y}$, $\boldsymbol{B}^{PD} = \boldsymbol{I} \in \ ^{n_y \times n_y}$, $\boldsymbol{C}^{PD}(h) = \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{C}_j^{PD}$, and $\boldsymbol{D}^{PD}(h) = \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{D}_j^{PD}$. The coefficients of the fuzzy PD controller can be obtained by

$$\begin{cases} \boldsymbol{K}_j^{P} = \boldsymbol{D}_j^{PD} - \tau\boldsymbol{C}_j^{PD} \\ \boldsymbol{K}_j^{D} = -\tau^2\boldsymbol{C}_j^{PD}. \end{cases} \tag{18}$$

Without any loss of generality, we only investigate the design of H_∞ fuzzy PID controllers in the following analysis. It is easy to verify that all the results are applicable to the design of fuzzy PI and PD controllers.

3.2. Transformation from fuzzy PID control system to fuzzy SOF control system

With the augmentation technique [27, 28], we can transform the fuzzy PID control system into the following fuzzy SOF control system

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{tmp}(h) & \boldsymbol{B}_{1tmp}(h) & \boldsymbol{B}_{2tmp}(h) \\ \boldsymbol{C}_{1tmp}(h) & \boldsymbol{D}_{1tmp}(h) & \boldsymbol{0} \\ \boldsymbol{C}_{2tmp}(h) & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \\ \boldsymbol{u} \end{bmatrix} \tag{19}$$

$$\boldsymbol{u} = \boldsymbol{F}_{tmp}(h)\boldsymbol{y} \tag{20}$$

where

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^T & \boldsymbol{x}_c^T \end{bmatrix}^T,\ \boldsymbol{y} = \begin{bmatrix} \boldsymbol{x}_c^T & \boldsymbol{y}^T \end{bmatrix}^T,\ \boldsymbol{u} = \begin{bmatrix} \boldsymbol{x}_c^T & \boldsymbol{u}^T \end{bmatrix}^T$$

$$\boldsymbol{A}_{tmp}(h) = \begin{bmatrix} \boldsymbol{A}(h) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{B}_{1tmp}(h) = \begin{bmatrix} \boldsymbol{B}_1(h) \\ \boldsymbol{0} \end{bmatrix},\ \boldsymbol{B}_{2tmp}(h) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{B}_2(h) \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{C}_{1tmp}(h) = \begin{bmatrix} \boldsymbol{C}_1(h) & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{D}_{1tmp}(h) = \boldsymbol{D}_1(h)$$

$$\boldsymbol{C}_{2tmp}(h) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}_2(h) & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{F}_{tmp}(h) = \begin{bmatrix} \boldsymbol{A}^{PID} & \boldsymbol{B}^{PID} \\ \boldsymbol{C}^{PID}(h) & \boldsymbol{D}^{PID}(h) \end{bmatrix}.$$

Note that the matrices $\boldsymbol{A}^{PID}$ and $\boldsymbol{B}^{PID}$ in $\boldsymbol{F}_{tmp}(h)$ are constant. Thus, they can be further extracted from (20). For this purpose, we partition the matrices $\boldsymbol{B}_{2tmp}(h)$ and $\boldsymbol{F}_{tmp}(h)$ as follows:

$$\boldsymbol{B}_{2tmp}(h) = \begin{bmatrix} \boldsymbol{B}_{2tmp}^1 & \boldsymbol{B}_{2tmp}^2(h) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix} & \begin{bmatrix} \boldsymbol{B}_2(h) \\ \boldsymbol{0} \end{bmatrix} \end{bmatrix} \tag{21}$$

$$\boldsymbol{F}_{tmp}(h) = \begin{bmatrix} \boldsymbol{F}_{known} \\ \boldsymbol{F}(h) \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{known} \\ \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{F}_j \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \boldsymbol{A}^{PID} & \boldsymbol{B}^{PID} \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{C}^{PID}(h) & \boldsymbol{D}^{PID}(h) \end{bmatrix} \end{bmatrix} \tag{22}$$

where $\boldsymbol{B}_{2tmp}^1 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}^T$, $\boldsymbol{B}_{2tmp}^2(h) = \begin{bmatrix} \boldsymbol{B}_2^T(h) & \boldsymbol{0} \end{bmatrix}^T$, $\boldsymbol{F}_{known} = \begin{bmatrix} \boldsymbol{A}^{PID} & \boldsymbol{B}^{PID} \end{bmatrix}$, and $\boldsymbol{F}_j = \begin{bmatrix} \boldsymbol{C}_j^{PID} & \boldsymbol{D}_j^{PID} \end{bmatrix}$. Moreover, we define a set of new matrices

$$\begin{aligned}
\boldsymbol{A}(h) = \sum_{i=1}^{r} h_i(\boldsymbol{x})\boldsymbol{A}_i &= \boldsymbol{A}_{tmp}(h) + \boldsymbol{B}_{2tmp}^1(h)\boldsymbol{F}_{known}\boldsymbol{C}_{2tmp}(h) \\
&= \begin{bmatrix} \boldsymbol{A}(h) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}^{PID} & \boldsymbol{B}^{PID} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}_2(h) & \boldsymbol{0} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{A}(h) & \boldsymbol{0} \\ \boldsymbol{B}^{PID}\boldsymbol{C}_2(h) & \boldsymbol{A}^{PID} \end{bmatrix} \\
&= \sum_{i=1}^{r} h_i(\boldsymbol{x}) \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{0} \\ \boldsymbol{B}^{PID}\boldsymbol{C}_{2i} & \boldsymbol{A}^{PID} \end{bmatrix}
\end{aligned}$$

$$\boldsymbol{B}_1(h) = \sum_{i=1}^{r} h_i(\boldsymbol{x})\boldsymbol{B}_{1i} = \boldsymbol{B}_{1tmp}(h) = \sum_{i=1}^{r} h_i(\boldsymbol{x}) \begin{bmatrix} \boldsymbol{B}_{1i} \\ \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{B}_2(h) = \sum_{i=1}^{r} h_i(x_1)\boldsymbol{B}_{2i} = \boldsymbol{B}_{2tmp}^2(h) = \sum_{i=1}^{r} h_i(x_1) \begin{bmatrix} \boldsymbol{B}_{2i} \\ \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{C}_1(h) = \sum_{i=1}^{r} h_i(x_1)\boldsymbol{C}_{1i} = \boldsymbol{C}_{1tmp}(h) = \sum_{j=1}^{r} h_i(x_1) \begin{bmatrix} \boldsymbol{C}_{1i} & \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{D}_1(h) = \sum_{i=1}^{r} h_i(x_1)\boldsymbol{D}_{1i} = \boldsymbol{D}_{1tmp}(h) = \sum_{i=1}^{r} h_i(x_1)\boldsymbol{D}_{1i}$$

$$\boldsymbol{C}_2(h) = \sum_{i=1}^{r} h_i(x_1)\boldsymbol{C}_{2i} = \boldsymbol{C}_{2tmp}(h) = \sum_{j=1}^{r} h_i(x_1)\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}_{2i} & \boldsymbol{0} \end{bmatrix}$$

where

$$\boldsymbol{A}_i = \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{0} \\ \boldsymbol{B}^{PID}\boldsymbol{C}_{2i} & \boldsymbol{A}^{PID} \end{bmatrix},\ \boldsymbol{B}_{1i} = \begin{bmatrix} \boldsymbol{B}_{1i} \\ \boldsymbol{0} \end{bmatrix},\ \boldsymbol{B}_{2i} = \begin{bmatrix} \boldsymbol{B}_{2i} \\ \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{C}_{1i} = \begin{bmatrix} \boldsymbol{C}_{1i} & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{D}_{1i} = \boldsymbol{D}_{1i}$$

$$\boldsymbol{C}_{2i} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}_{2i} & \boldsymbol{0} \end{bmatrix}.$$

From (19), we obtain

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{A}_{tmp}(h)\boldsymbol{x} + \boldsymbol{B}_{1tmp}(h)\boldsymbol{w} + \boldsymbol{B}_{2tmp}(h)\boldsymbol{u} \\ &= \boldsymbol{A}_{tmp}(h)\boldsymbol{x} + \boldsymbol{B}_{1tmp}(h)\boldsymbol{w} + \begin{bmatrix} \boldsymbol{B}^1_{2tmp}(h) & \boldsymbol{B}^2_{2tmp}(h) \end{bmatrix}\begin{bmatrix} \boldsymbol{F}_{known} \\ \boldsymbol{F}(h) \end{bmatrix}\boldsymbol{C}_{2tmp}(h)\boldsymbol{x} \\ &= (\boldsymbol{A}_{tmp}(h) + \boldsymbol{B}^1_{2tmp}(h)\boldsymbol{F}_{known}\boldsymbol{C}_{2tmp}(h))\boldsymbol{x} + \boldsymbol{B}_{1tmp}(h)\boldsymbol{w} + \boldsymbol{B}^2_{2tmp}(h) \\ &\quad \times \boldsymbol{F}(h)\boldsymbol{C}_{2tmp}(h)\boldsymbol{x} \\ &= \boldsymbol{A}(h)\boldsymbol{x} + \boldsymbol{B}_1(h)\boldsymbol{w} + \boldsymbol{B}_2(h)\boldsymbol{u} \end{aligned} \tag{23}$$

$$\boldsymbol{z} = \boldsymbol{C}_{1tmp}(h)\boldsymbol{x} + \boldsymbol{D}_{1tmp}(h)\boldsymbol{w} = \boldsymbol{C}_1(h)\boldsymbol{x} + \boldsymbol{D}_1(h)\boldsymbol{w} \tag{24}$$

$$\boldsymbol{y} = \boldsymbol{C}_{2tmp}(h)\boldsymbol{x} = \boldsymbol{C}_2(h)\boldsymbol{x}. \tag{25}$$

Rewriting (23)-(25) in a more compact form, we have

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}(h) & \boldsymbol{B}_1(h) & \boldsymbol{B}_2(h) \\ \boldsymbol{C}_1(h) & \boldsymbol{D}_1(h) & \boldsymbol{0} \\ \boldsymbol{C}_2(h) & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \\ \boldsymbol{u} \end{bmatrix}. \tag{26}$$

In addition, the new fuzzy SOF controller is described by

$$\boldsymbol{u} = \boldsymbol{F}(h)\boldsymbol{y} \tag{27}$$

where $\boldsymbol{F}(h) = \sum_{j=1}^{r} h_j(\boldsymbol{x})\boldsymbol{F}_j$ and $\boldsymbol{F}_j = \begin{bmatrix} \boldsymbol{C}^{PID}_j & \boldsymbol{D}^{PID}_j \end{bmatrix}$.

For the design specifications of the fuzzy PID control systems, we have the following theorem.

Theorem 1: Consider the fuzzy PID control system given by the T-S fuzzy model (3) and the fuzzy PID controller (14) as well as the fuzzy SOF control system given by the augmented T-S fuzzy model (26) and the fuzzy SOF controller (27). These two kinds of control systems are identical in the sense of stability characteristic and H_∞ performance.

Proof. For stability characteristic, we assume that $\boldsymbol{w} \equiv 0$. Substituting (14) into (3), we have

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}(h) + \boldsymbol{B}_2(h)\boldsymbol{D}^{PID}(h)\boldsymbol{C}_2(h) & \boldsymbol{B}_2(h)\boldsymbol{C}^{PID}(h) \\ \boldsymbol{B}^{PID}\boldsymbol{C}_2(h) & \boldsymbol{A}^{PID} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{bmatrix}. \tag{28}$$

On the other hand, from (26) and (27), we obtain

$$\begin{aligned} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{bmatrix} &= \boldsymbol{A}(h)\boldsymbol{x} + \boldsymbol{B}_2(h)\boldsymbol{F}(h)\boldsymbol{C}_2(h)\boldsymbol{x} \\ &= \begin{bmatrix} \boldsymbol{A}(h) & \boldsymbol{0} \\ \boldsymbol{B}^{PID}\boldsymbol{C}_2(h) & \boldsymbol{A}^{PID} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_2(h) \\ \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}^{PID}(h) & \boldsymbol{D}^{PID}(h) \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}_2(h) & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}(h) + \boldsymbol{B}_2(h)\boldsymbol{D}^{PID}(h)\boldsymbol{C}_2(h) & \boldsymbol{B}_2(h)\boldsymbol{C}^{PID}(h) \\ \boldsymbol{B}^{PID}\boldsymbol{C}_2(h) & \boldsymbol{A}^{PID} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{bmatrix} \end{aligned}$$

which is identical to (28). It reveals that the T-S fuzzy PID control system and the fuzzy SOF control system have the same stability characteristic. As for H_∞ performance, note that in the two control systems, the channels from $\boldsymbol{w}$ to $\boldsymbol{z}$ are equivalent to each other since $\boldsymbol{z} = \boldsymbol{C}_1(h)\boldsymbol{x} + \boldsymbol{D}_1(h)\boldsymbol{w} = \boldsymbol{C}_1(h)\boldsymbol{x} + \boldsymbol{D}_1(h)\boldsymbol{w}$. □

Next, we will present a fuzzy SOF controller design method based on the ILMI method [31].

3.3. An ILMI algorithm for fuzzy SOF controller design

In the design of fuzzy SOF controllers, we often have to solve the bilinear matrix inequalities (BMIs), which are nonconvex and NP-hard [28, 32-35]. In this subsection, we use the ILMI method to solve the BMIs. First, we give the following theorem.

Theorem 2: The fuzzy SOF control system given by (26) and (27) is asymptotically stable with the prescribed H_∞ performance $\gamma > 0$, if there exist matrices $\boldsymbol{P} > 0$, $\boldsymbol{X} > 0$, and $\boldsymbol{F}_j$, $j = 1, 2, \quad , r$, such that

$$\boldsymbol{\Xi}_{ii} < 0,\ i = 1, 2, \quad , r \tag{29}$$

$$\frac{2}{r-1}\boldsymbol{\Xi}_{ii} + \boldsymbol{\Xi}_{ij} + \boldsymbol{\Xi}_{ji} < 0,\ 1 \le i < j \le r \tag{30}$$

where

$$\boldsymbol{\Xi}_{ij} = \begin{bmatrix} \boldsymbol{A}_i^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_i - \boldsymbol{P}\boldsymbol{B}_{2i}\boldsymbol{B}_{2j}^T\boldsymbol{X} - \boldsymbol{X}\boldsymbol{B}_{2i}\boldsymbol{B}_{2j}^T\boldsymbol{P} + \boldsymbol{X}\boldsymbol{B}_{2i}\boldsymbol{B}_{2j}^T\boldsymbol{X} & * & * & * \\ \boldsymbol{B}_{1i}^T\boldsymbol{P} & -\gamma^2\boldsymbol{I} & * & * \\ \boldsymbol{C}_{1i} & \boldsymbol{D}_{1i} & -\boldsymbol{I} & * \\ \boldsymbol{B}_{2i}\boldsymbol{P} + \boldsymbol{F}_j\boldsymbol{C}_{2i} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix}. \tag{31}$$

Proof. From (26) and (27), the dynamics of the fuzzy SOF control system can be described by

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}(h) + \boldsymbol{B}_2(h)\boldsymbol{F}(h)\boldsymbol{C}_2(h) & \boldsymbol{B}_1(h) \\ \boldsymbol{C}_1(h) & \boldsymbol{D}_1(h) \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \end{bmatrix}. \tag{32}$$

Given a quadratic Lyapunov function $V = \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P} > 0$, its time derivative along with the trajectory of the closed-loop system (32) is as follows:

$$\begin{aligned} V &= \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x} \\ &= \boldsymbol{x}^T (\boldsymbol{A}(h) + \boldsymbol{B}_2(h)\boldsymbol{F}(h)\boldsymbol{C}_2(h))^T \boldsymbol{P}\boldsymbol{x} + \boldsymbol{w}^T \boldsymbol{B}_1^T(h)\boldsymbol{P}\boldsymbol{x} \\ &\quad + \boldsymbol{x}^T \boldsymbol{P}(\boldsymbol{A}(h) + \boldsymbol{B}_2(h)\boldsymbol{F}(h)\boldsymbol{C}_2(h))\boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{B}_1(h)\boldsymbol{w}. \end{aligned} \tag{33}$$

A sufficient condition in which the SOF control system (32) is asymptotically stable with the prescribed H_∞ performance $\gamma > 0$ is (see [28])

$$V + \boldsymbol{z}^T \boldsymbol{z} - \gamma^2 \boldsymbol{w}^T \boldsymbol{w} < 0. \tag{34}$$

Substituting (32) and (33) into (34), we obtain

$$\begin{aligned} & V + \boldsymbol{z}^T \boldsymbol{z} - \gamma^2 \boldsymbol{w}^T \boldsymbol{w} \\ &= \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{A}^T(h)\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}(h) + \boldsymbol{B}_2(h)\boldsymbol{F}(h)\boldsymbol{C}_2(h)\boldsymbol{P} & \\ \quad + \boldsymbol{P}\boldsymbol{C}_2^T(h)\boldsymbol{F}^T(h)\boldsymbol{B}_2^T(h) + \boldsymbol{C}_1^T(h)\boldsymbol{C}_1(h) & * \\ \boldsymbol{B}_1^T(h)\boldsymbol{P} + \boldsymbol{D}_1^T(h)\boldsymbol{C}_1(h) & -\gamma^2 \boldsymbol{I} + \boldsymbol{D}_1^T(h)\boldsymbol{D}_1(h) \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \end{bmatrix} \\ &< 0. \end{aligned} \tag{35}$$

Furthermore, the inequality (35) is equivalent to

$$\begin{bmatrix} \boldsymbol{A}^T(h)\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}(h) + \boldsymbol{B}_2(h)\boldsymbol{F}(h)\boldsymbol{C}_2(h)\boldsymbol{P} & \\ \quad + \boldsymbol{P}\boldsymbol{C}_2^T(h)\boldsymbol{F}^T(h)\boldsymbol{B}_2^T(h) + \boldsymbol{C}_1^T(h)\boldsymbol{C}_1(h) & * \\ \boldsymbol{B}_1^T(h)\boldsymbol{P} + \boldsymbol{D}_1^T(h)\boldsymbol{C}_1(h) & -\gamma^2 \boldsymbol{I} + \boldsymbol{D}_1^T(h)\boldsymbol{D}_1(h) \end{bmatrix} < 0. \tag{36}$$

Applying the Schur Complement to (36), we have

$$\begin{bmatrix} \boldsymbol{A}^T(h)\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}(h) + \boldsymbol{B}_2(h)\boldsymbol{F}(h)\boldsymbol{C}_2(h)\boldsymbol{P} + \boldsymbol{P}\boldsymbol{C}_2^T(h)\boldsymbol{F}^T(h)\boldsymbol{B}_2^T(h) & * & * \\ \boldsymbol{B}_1^T(h)\boldsymbol{P} & -\gamma^2 \boldsymbol{I} & * \\ \boldsymbol{C}_1(h) & \boldsymbol{D}_1(h) & -\boldsymbol{I} \end{bmatrix} < 0. \tag{37}$$

Since $\boldsymbol{C}_1^T(h)\boldsymbol{F}^T(h)\boldsymbol{F}(h)\boldsymbol{C}_1(h) > 0$, inequality (37) can be guaranteed by

$$
\begin{bmatrix}
A^T(h)P + PA(h) + B_2(h)F(h)C_2(h)P + PC_2^T(h)F^T(h)B_2^T(h) + C_1^T(h)F^T(h)F(h)C_1(h) & * & * \\
B_1^T(h)P & -\gamma^2 I & * \\
C_1(h) & D_1(h) & -I
\end{bmatrix}
$$

$$
= \begin{bmatrix}
A^T(h)P + PA(h) - PB_2(h)B_2^T(h)P + (B_2^T(h)P + F(h)C_2(h))^T(B_2^T(h)P + F(h)C_2(h)) & * & * \\
B_1^T(h)P & -\gamma^2 I & * \\
C_1(h) & D_1(h) & -I
\end{bmatrix} < 0. \tag{38}
$$

Applying the Schur Complement to (38) again, we have

$$
\begin{bmatrix}
A^T(h)P + PA(h) - PB_2(h)B_2^T(h)P & * & * & * \\
B_1^T(h)P & -\gamma^2 I & * & * \\
C_1(h) & D_1(h) & -I & * \\
B_2^T(h)P + F(h)C_2(h) & 0 & 0 & -I
\end{bmatrix} < 0 \cdot \tag{39}
$$

To accommodate the $-PB_2(h)B_2^T(h)P$ term, the inequality $-PB_2(h)B_2^T(h)P \le -PB_2(h)B_2^T(h) \times X - XB_2(h)B_2^T(h)P + XB_2(h)B_2^T(h)X$ (derived by $(P - X)B_2(h)B_2^T(h)(P - X) \ge 0$) is used in (39). Therefore, inequality (39) can be guaranteed by

$$
\begin{bmatrix}
A^T(h)P + PA(h) - PB_2(h)B_2^T(h)X - XB_2(h)B_2^T(h)P + XB_2(h)B_2^T(h)X & * & * & * \\
B_1^T(h)P & -\gamma^2 I & * & * \\
C_1(h) & D_1(h) & -I & * \\
B_2^T(h)P + F(h)C_2(h) & 0 & 0 & -I
\end{bmatrix} < 0 \cdot \tag{40}
$$

The left hand side of (39) is the fuzzy summation $\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(x)h_j(x)\Xi_{ij}$, where Ξ_{ij} is defined in (31). Based on the parameterized LMI method [36], conditions (29)-(31) are sufficient to assure the equality (40) holds. □

Conditions (29)-(31) of Theorem 2 are BMIs, which cannot be solved via existing convex optimization methods [37], since there exist products of P and X. In fact, conditions (29)-(31) are reduced to LMIs with respect to P and F_j, $j = 1, 2, \ldots, r$, when X is fixed. Furthermore, if there exists a feasible solution (for instance, P and F_j, $j = 1, 2, \ldots, r$) to conditions (29)-(31) with a fixed X, F_j, $j = 1, 2, \ldots, r$ are the desired static feedback gains. However, conditions (29)-(31) may probably have no feasible solutions for an arbitrarily fixed X. Therefore, we present the following iterative algorithm to find a suitable fixed X such that conditions (29)-(31) may have feasible solutions.

Algorithm 1.

Step 1): Set iterative number $k = 1$ *and select initial value* $\boldsymbol{X}_k = \lambda \boldsymbol{I}$, *where* $\lambda > 0$.

Step 2): Solve the following generalized eigenvalue minimization problem for $\boldsymbol{P}_k > 0$ *and* $\boldsymbol{F}_j$.

P1: *Minimize* α_k *subject to the LMI constraints*

$$\boldsymbol{\Omega}_{ii} < 0,\ i = 1, 2, \quad , r \tag{41}$$

$$\frac{2}{r-1}\boldsymbol{\Omega}_{ii} + \boldsymbol{\Omega}_{ij} + \boldsymbol{\Omega}_{ji} < 0,\ 1 \le i < j \le r \tag{42}$$

where

$$\boldsymbol{\Omega}_{ij} = \begin{bmatrix} \boldsymbol{A}_i^T \boldsymbol{P}_k + \boldsymbol{P}_k \boldsymbol{A}_i - \boldsymbol{P}_k \boldsymbol{B}_{2i} \boldsymbol{B}_{2j}^T \boldsymbol{X}_k - \boldsymbol{X}_k \boldsymbol{B}_{2i} \boldsymbol{B}_{2j}^T \boldsymbol{P}_k + \boldsymbol{X}_k \boldsymbol{B}_{2i} \boldsymbol{B}_{2j}^T \boldsymbol{X}_k - \alpha_k \boldsymbol{P}_k & * & * & * \\ \boldsymbol{B}_{1i}^T \boldsymbol{P}_k & -\gamma^2 \boldsymbol{I} & * & * \\ \boldsymbol{C}_{1i} & \boldsymbol{D}_{1i} & -\boldsymbol{I} & * \\ \boldsymbol{B}_{2i} \boldsymbol{P}_k + \boldsymbol{F}_j \boldsymbol{C}_{2i} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} \tag{43}$$

and $\boldsymbol{X}_k$ *is known. Denote* α_k^* *as the minimized value of* α_k. *If* $\alpha_k^* \le 0$, $\boldsymbol{F}_j$, $j = 1, 2, \quad , r$ *are the desired fuzzy SOF gain matrices, stop.*

Step 3): Solve the following objective function minimization problem for $\boldsymbol{P}_k > 0$ *and* $\boldsymbol{F}_j$.

P2: *Minimize* $trace\{\boldsymbol{P}_k\}$ *subject to LMI constraints (44) and (45) for* $\boldsymbol{P}_k$ *and* $\boldsymbol{F}_j$, *where* $\alpha_k = \alpha_k^*$ *and* $\boldsymbol{X}_k$ *are known.*

Step 4): Denote $\boldsymbol{P}_k^*$ *as the solution of P2. If* $\left\| \boldsymbol{X}_k - \boldsymbol{P}_k^* \right\| \le \delta$, *where* δ *is a prescribed tolerance, then the algorithm cannot get a feasible solution, stop; else, set* $k = k + 1$ *and* $\boldsymbol{X}_k = \boldsymbol{P}_{k-1}^*$, *go to step 2.*

Remark 3: As mentioned above, conditions (29)-(31) may have no solutions for an arbitrarily fixed $\boldsymbol{X}$. Therefore, we subtract the matrix $\alpha_k \boldsymbol{\Pi}_k = \alpha_k diag\{\boldsymbol{P}_k, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}\}$ from $\boldsymbol{\Xi}_{ij}$ in (31) (see (43)). The optimization problem P1 in Step 2, which is an generalized eigenvalue minimization problem, guarantees that conditions (41) and (42) have a solution ($\boldsymbol{P}_k^*$ and $\boldsymbol{F}_j^*$, $j = 1, 2, \quad , r$) for the fixed $\boldsymbol{X}_k$ and $\alpha_k = \alpha_k^*$. If $\alpha_k^* \le 0$, the solution is also a solution of conditions (29) and (31) with $\boldsymbol{X} = \boldsymbol{X}_k$. To see this point, let $\boldsymbol{P} = \boldsymbol{P}_k^*$ and $\boldsymbol{F}_j = \boldsymbol{F}_j^*$, $j = 1, 2, \quad , r$, then it follows from (41)-(43) that

$$\boldsymbol{\Omega}_{ii} = \boldsymbol{\Xi}_{ii} - \alpha_k^* \boldsymbol{\Pi}_k < 0,\ i = 1, 2, \quad , r$$

$$\frac{2}{r-1}\boldsymbol{\Omega}_{ii} + \boldsymbol{\Omega}_{ij} + \boldsymbol{\Omega}_{ji} = \frac{2}{r-1}\boldsymbol{\Xi}_{ii} + \boldsymbol{\Xi}_{ij} + \boldsymbol{\Xi}_{ji} - \frac{2\alpha_k^* r}{r-1}\boldsymbol{\Pi}_k < 0,\ 1 \le i < j \le r.$$

Transposing the $\boldsymbol{\Pi}_k$ term to the right of the above inequalities, we have

$$\boldsymbol{\Xi}_{ii} < \alpha_k^* \boldsymbol{\Pi}_k \le 0,\ i = 1, 2, \quad , r$$

$$\frac{2}{r-1}\boldsymbol{\Xi}_{ii} + \boldsymbol{\Xi}_{ij} + \boldsymbol{\Xi}_{ji} < \frac{2\alpha_k^* r}{r-1}\boldsymbol{\Pi}_k \le 0,\ 1 \le i < j \le r.$$

Obviously, $\boldsymbol{P}_k^*$ and $\boldsymbol{F}_j^*$, $j = 1, 2, \quad , r$ constitute a solution to conditions (29)-(31) with $\boldsymbol{X} = \boldsymbol{X}_k$. $\boldsymbol{F}_j^*$, $j = 1, 2, \quad , r$ are the desired static output feedback gains, and we can terminate the algorithm. If $\alpha_k^* > 0$, we continue. The problem P1 guarantees the progressive reduction of α_k^*. In other words, a solution $\alpha_{k+1}^* \le \alpha_k^*$ can be found in Step 2 since conditions (41)-(43) are feasible with $\boldsymbol{P}_{k+1} = \boldsymbol{P}_k^*$ and $\alpha_{k+1} = \alpha_k^*$ (see [31] and [32]).

Remark 4: The optimization problem P2 in Step 3 is necessary to guarantee the convergence of the algorithm. For a fixed α_k, we can see that conditions (41)-(43) are feasible with $\boldsymbol{P}_{k+1} = \boldsymbol{P}_k^*$. It means that the sequence $trace\{\boldsymbol{P}_k^*\}$ is monotonically decreasing. As a result, the algorithm is convergent no matter whether we could find a fixed $\boldsymbol{X}$ for the feasibility of conditions (29)-(31) or not. It is worth mentioning that the proposed algorithm is not a global optimization method for the BMIs (29)-(31). If the algorithm fails to yield any solutions, it does not mean that feasible solutions do not exist for the BMIs.

3.4. Fuzzy PID controller design procedure

When the gain matrices $\boldsymbol{F}_j$, $j = 1, 2, \quad , r$ of the fuzzy SOF controller are derived, the fuzzy PID controller can be recovered via (13). To summarize, we present the following fuzzy PID controller design procedure. We need to emphasize that the following design procedure is also applicable to the fuzzy PI and PD controller design.

Procedure 1.

Step 1)*: Represent the fuzzy PID controller by the state space realization (13).*

Step 2)*: Transform the fuzzy PID control system given by (3) and (14) into the fuzzy SOF control system given by (26) and (27).*

Step 3)*: Derive the gain matrices* $\boldsymbol{F}_j$, $j = 1, 2, \quad , r$ *of the fuzzy SOF controller by Algorithm 1.*

Step 4)*: Recover the fuzzy PID controller from the fuzzy SOF controller via (13).*

4. Simulation examples

Example 1: The well-known inverted pendulum example is used in this example. The dynamics of the pendulum is as follows:

$$\begin{cases} x_1 = x_2 \\ x_2 = \dfrac{g\sin(x_1) - amlx_2^2\sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)} + 0.01w \\ z = \boldsymbol{C}_1\boldsymbol{x} + D_1 w \\ y = \boldsymbol{C}_2\boldsymbol{x} \end{cases} \tag{44}$$

where $g = 9.8m/s^2$, $M = 8kg$, $m = 2kg$, $a = 1/(M+m) = 0.1$, $l = 0.5m$, x_1 is the angular of the pendulum, x_2 is the angular velocity, and

$$\boldsymbol{C}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix},\ D_1 = 0.1,\ \boldsymbol{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The nonlinear plant can be described by the following two-rule T-S fuzzy model [38, 39]

$$\begin{cases} \boldsymbol{x} = \sum_{i=1}^{2} h_1(x_1)(\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{B}_{1i}w + \boldsymbol{B}_{2i}u) \\ z = \sum_{i=1}^{2} h_1(x_1)(\boldsymbol{C}_{1i}\boldsymbol{x} + D_{1i}w) \\ y = \sum_{i=1}^{2} h_1(x_1)\boldsymbol{C}_{2i}\boldsymbol{x} \end{cases} \tag{45}$$

where $x_1 \in \begin{bmatrix} -\pi/3 & \pi/3 \end{bmatrix}$ and

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix},\ \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 12.6305 & 0 \end{bmatrix}$$

$$\boldsymbol{B}_{11} = \boldsymbol{B}_{12} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix},\ \boldsymbol{B}_{21} = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix},\ \boldsymbol{B}_{22} = \begin{bmatrix} 0 \\ -0.0779 \end{bmatrix}$$

$$\boldsymbol{C}_{11} = \boldsymbol{C}_{12} = \begin{bmatrix} 1 & 1 \end{bmatrix},\ D_{11} = D_{12} = 0.1,\ \boldsymbol{C}_{21} = \boldsymbol{C}_{22} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The membership functions are defined as follows:

$$h_1(x_1) = (1 - (1 + e^{-7(x_1 - \pi/4)})^{-1})(1 + e^{-7(x_1 + \pi/4)})^{-1}$$

$$h_2(x_1) = 1 - h_1(x_1).$$

The fuzzy PID controller, whose fuzzy rules are shown in (5), is implemented here to control the inverted pendulum. Assuming that the H_∞ performance $\gamma = 1$, we will use Procedure 1 to design the fuzzy PID controller.

Step 1): Let $\tau = 0.1$, the state space realization of the fuzzy PID controller can be obtained as follows:

$$\begin{bmatrix} \dot{\boldsymbol{x}}_c \\ u \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^{PID} & \boldsymbol{B}^{PID} \\ \boldsymbol{C}^{PID}(h) & D^{PID}(h) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_c \\ y \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^{PID} & \boldsymbol{B}^{PID} \\ \sum_{j=1}^{2} h_j(x_1)\boldsymbol{C}_j^{PID} & \sum_{j=1}^{2} h_j(x_1)D_j^{PID} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_c \\ y \end{bmatrix} \tag{46}$$

where $\boldsymbol{C}_j^{PID}$ and D_j^{PID} are matrices that need to be determined, and

$$\boldsymbol{A}^{PID} = \begin{bmatrix} -\tau^{-1} & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ 1 & 0 \end{bmatrix},\ \boldsymbol{B}^{PID} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Step 2): Transform the fuzzy PID control system given by (45) and (46) into the following fuzzy SOF control system, where the T-S fuzzy model is

$$\begin{cases} \dot{\boldsymbol{x}} = \sum_{i=1}^{2} h_i(x_1)(\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{B}_{1i}w + \boldsymbol{B}_{2i}u) \\ z = \sum_{i=1}^{2} h_i(x_1)(\boldsymbol{C}_{1i}\boldsymbol{x} + D_{1i}w) \\ \boldsymbol{y} = \sum_{i=1}^{2} h_i(x_1)\boldsymbol{C}_{2i}\boldsymbol{x} \end{cases} \tag{47}$$

where $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^T & \boldsymbol{x}_c^T \end{bmatrix}^T$, $\boldsymbol{y} = \begin{bmatrix} \boldsymbol{x}_c^T & y \end{bmatrix}^T$, and

$$\boldsymbol{A}_1 = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{0} \\ \boldsymbol{B}^{PID}\boldsymbol{C}_{21} & \boldsymbol{A}^{PID} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 17.2941 & 0 & 0 & 0 \\ 1 & 0 & -10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} \boldsymbol{A}_2 & \boldsymbol{0} \\ \boldsymbol{B}^{PID}\boldsymbol{C}_{22} & \boldsymbol{A}^{PID} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 12.6305 & 0 & 0 & 0 \\ 1 & 0 & -10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{B}_{11} = \boldsymbol{B}_{12} = \begin{bmatrix} \boldsymbol{B}_{11} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} 0 & 0.01 & 0 & 0 \end{bmatrix}^T$$

$$\boldsymbol{B}_{21} = \begin{bmatrix} \boldsymbol{B}_{21} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} 0 & -0.1765 & 0 & 0 \end{bmatrix}^T$$

$$\boldsymbol{B}_{22} = \begin{bmatrix} \boldsymbol{B}_{22} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} 0 & -0.0779 & 0 & 0 \end{bmatrix}^T$$

$$\boldsymbol{C}_{11} = \boldsymbol{C}_{12} = \begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$

$$D_{11} = D_{12} = D_{11} = 0.1$$

$$C_{21} = C_{22} = \begin{bmatrix} \mathbf{0} & I \\ C_{2i} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and the SOF controller is

$$u = \sum\nolimits_{j=1}^{2} h_j(x_1) F_j y \tag{48}$$

where $F_j = \begin{bmatrix} C_j^{PID} & D_j^{PID} \end{bmatrix}$. Substituting (48) into (47) yields the dynamics of the fuzzy SOF control system

$$\begin{cases} x = \sum\nolimits_{i=1}^{2} h_i(x_1)((A_i + B_{2i} F_j) x + B_{1i} w) \\ z = \sum\nolimits_{i=1}^{2} h_i(x_1)(C_{1i} x + D_{1i} w). \end{cases}$$

Step 3): Set $\lambda = 10^4$ in Algorithm 1. By this algorithm, we derive the fuzzy SOF control laws as follows:

$$F_1 = \begin{bmatrix} -4515.0672 & 29.8664 & 640.5054 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -13858.3382 & 56.4146 & 1723.3647 \end{bmatrix}.$$

Step 4): From (13), we obtain the coefficients of the fuzzy PID controller as follows:

$$K_1^P = 188.7000,\ K_1^I = 2.9866,\ K_1^D = 45.1805$$

$$K_2^P = 336.9668,\ K_2^I = 5.6415,\ K_2^D = 138.6398.$$

To show the effectiveness of the obtained fuzzy PID controller, simulations have been carried out for the inverted pendulum system. Figure 2 depicts the responses of x_1, x_2, and u under initial conditions $x_0 = \begin{bmatrix} \pi/4 & 0 \end{bmatrix}^T$ and $w \equiv 0$, where we can observe that the fuzzy PID controller stabilizes the pendulum.

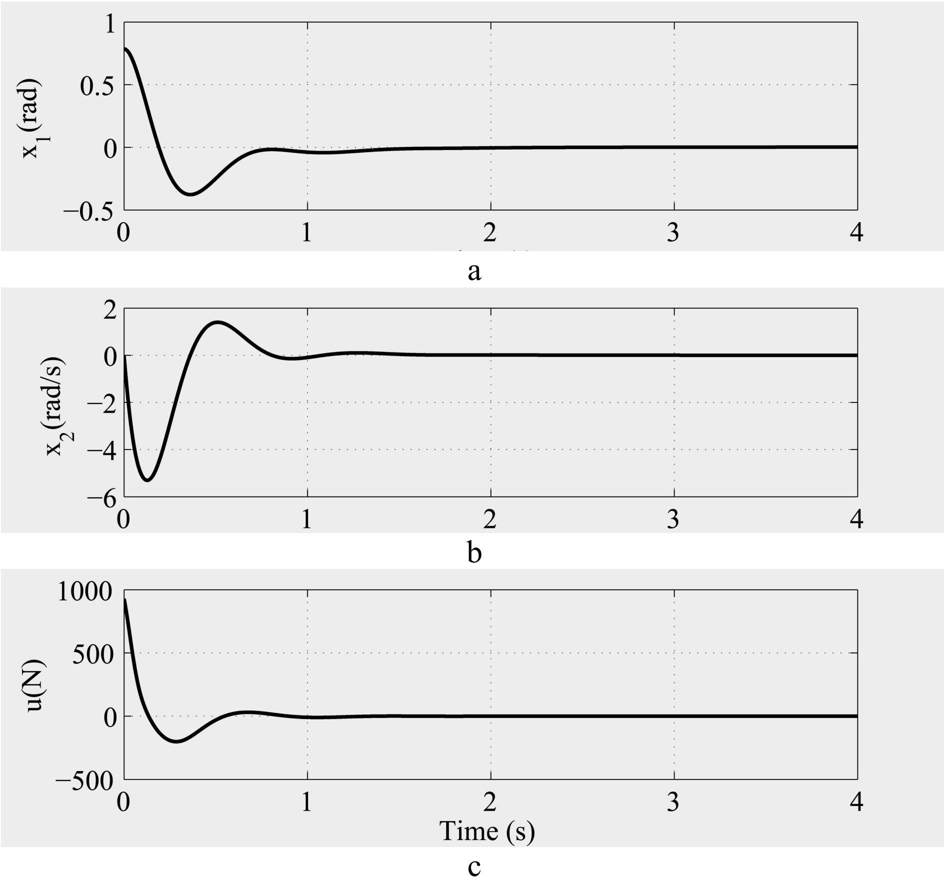


Fig. 2. *Responses of* x_1*,* x_2*, and* u *in Example 1 under initial conditions* $\boldsymbol{x}_0 = \left[\pi / 4 \;\; 0\right]^T$
a Response of x_1
b Response of x_2
c Response of u

Figure 3 demonstrates the response of the ratio $\sqrt{\int_0^t z^T z dt} / \sqrt{\int_0^t w^T w dt}$ under zero initial conditions and $w = 100e^{-t}\sin(2t)$. We can discover that the ratio is smaller than the H_∞ performance $\gamma = 1$. The results show that the designed fuzzy PID controller achieves the prescribed H_∞ performance.

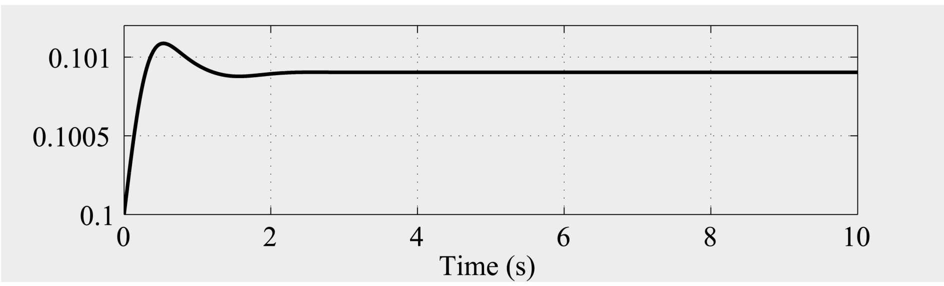
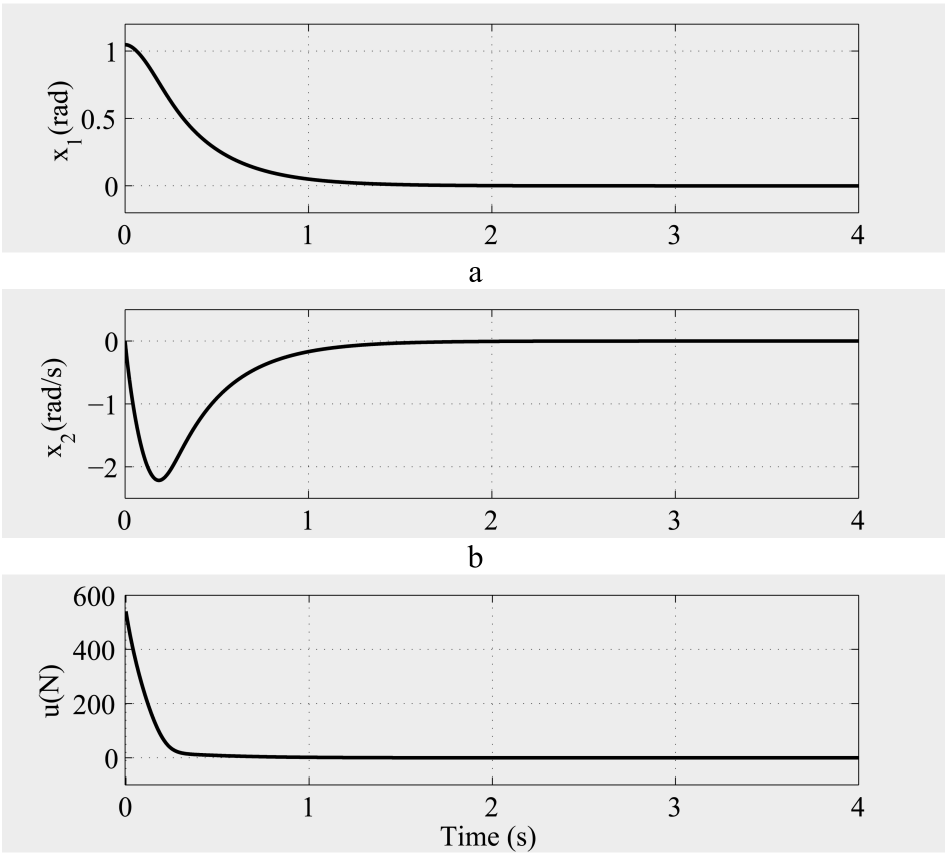


Fig. 3. *Response of the ratio* $\sqrt{\int_0^t z^T z dt} / \sqrt{\int_0^t w^T w dt}$ *in Example 1*

Example 2: The T-S fuzzy model (45) of the inverted pendulum (44) in Example 1 is used again in this example, where $\boldsymbol{C}_2 = \begin{bmatrix} 3.5 & 1 \end{bmatrix}$ and the other system matrices are the same as that of Example 1. The objective is to design a fuzzy PI controller ($K_j^D = 0$, $j = 1, 2, \ldots, r$ in (5)) such that the closed-loop is asymptotically stable with the prescribed H_∞ performance $\gamma = 1$. With Procedure 1, we derive the coefficients of the fuzzy PI controller as follows: $K_1^P = 250.5400$, $K_1^I = 1.1846$, $K_2^P = 130.9059$, and $K_2^I = 0.6311$ by setting $\lambda = 10^3$ in Algorithm 1. The responses of x_1, x_2, and u are shown in Fig. 4, where we can see that the closed-loop is stable under initial conditions $\boldsymbol{x}_0 = \begin{bmatrix} \pi/3 & 0 \end{bmatrix}^T$. Figure 5 illustrates the ratio response under zero initial conditions and $w = 100e^{-t}\sin(2t)$. Obviously, the ratio is much smaller than $\gamma = 1$.



c

Fig. 4. *Responses of x_1, x_2, and u in Example 2 under initial conditions $\boldsymbol{x}_0 = \begin{bmatrix} \pi/3 & 0 \end{bmatrix}^T$*
a Response of x_1
b Response of x_2
c Response of u

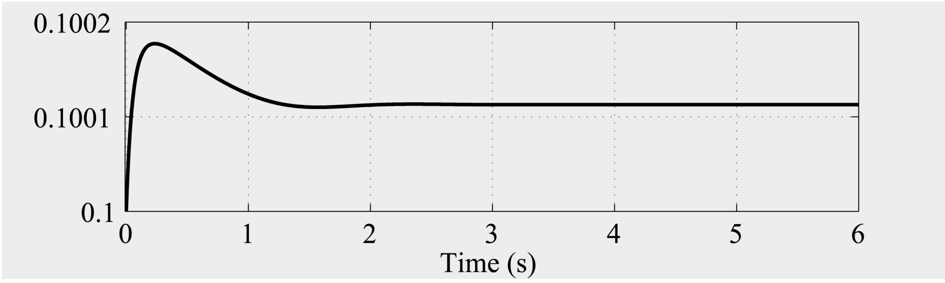


Fig. 5. *Response of the ratio* $\sqrt{\int_0^t z^T z dt} / \sqrt{\int_0^t w^T w dt}$ *in Example 2*

Example 3: In this example, we design a fuzzy PD controller for the Duffing forced-oscillation system [29], whose dynamics is as follows:

$$\begin{cases} x_1 = x_2 \\ x_2 = -x_1^3 + x_1 - 0.2x_2 + 0.1w + 0.3\cos t + u \\ z = x_1 + x_2 + 0.1w \\ y = x_1. \end{cases} \tag{49}$$

Let $u = 0.3\cos t + u$ and assume that $x_1 \in \begin{bmatrix} -d & d \end{bmatrix}$, where $d = 50$, the Duffing forced-oscillation system (49) can be described by the following two-rule T-S fuzzy model

$$\begin{cases} \boldsymbol{x} = \sum_{i=1}^{2} h_1(x_1)(\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{B}_{1i} w + \boldsymbol{B}_{2i} u) \\ z = \sum_{i=1}^{2} h_1(x_1)(\boldsymbol{C}_{1i} \boldsymbol{x} + D_{1i} w) \\ y = \sum_{i=1}^{2} h_1(x_1)\boldsymbol{C}_{2i} \boldsymbol{x} \end{cases} \tag{50}$$

where $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix}, \ \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 1-d^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -249 & -0.2 \end{bmatrix}$$

$$\boldsymbol{B}_{11} = \boldsymbol{B}_{12} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \ \boldsymbol{B}_{21} = \boldsymbol{B}_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{C}_{11} = \boldsymbol{C}_{12} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \ D_{11} = D_{12} = 0.1, \ \boldsymbol{C}_{21} = \boldsymbol{C}_{22} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The membership functions are

$$h_1(x_1) = 1 - \frac{x_1^2}{d^2}, \ h_2(x_1) = 1 - h_1(x_1) = \frac{x_1^2}{d^2}.$$

Assume that the H_∞ performance $\gamma = 1.2$. Using Procedure 1 with $\tau = 0.01$ and $\lambda = 10^3$, the coefficients of the fuzzy PD controller can be derived: $K_1^P = -873.6163$, $K_1^D = -3.2992$, $K_2^P = -623.9585$, and $K_2^D = -3.3004$. Applying the designed fuzzy PD controller to stabilize the Duffing forced-oscillation system (49), the responses of x_1, x_2, and u can be simulated and shown in Fig. 6, where the control signal u is added at $t \geq 40s$ (means that $u = u + 0.3\cos t = 0.3\cos t$ in the time interval $t \in [0 \quad 40]$). It can be seen from the top and middle subfigures of Fig. 6 that the controlled nonlinear system exhibits chaotic behavior when $0s \leq t \leq 40s$. When the control signal is applied for $t \geq 40s$, the system is stabilized within about 6 seconds (from $t = 40s$ to $46s$). The detailed responses of x_1, x_2, and u in the time interval $t \in [40 \quad 46]$ are depicted in the upper right corner of the three subfigures. When the states of the nonlinear system are stabilized at the origin, the control signal u has to equal to $-0.3\cos t$ in order to compensate the term $0.3\cos t$. The lower right corner of the bottom subfigure c shows the response of control signal u for $46s \leq t \leq 60s$.

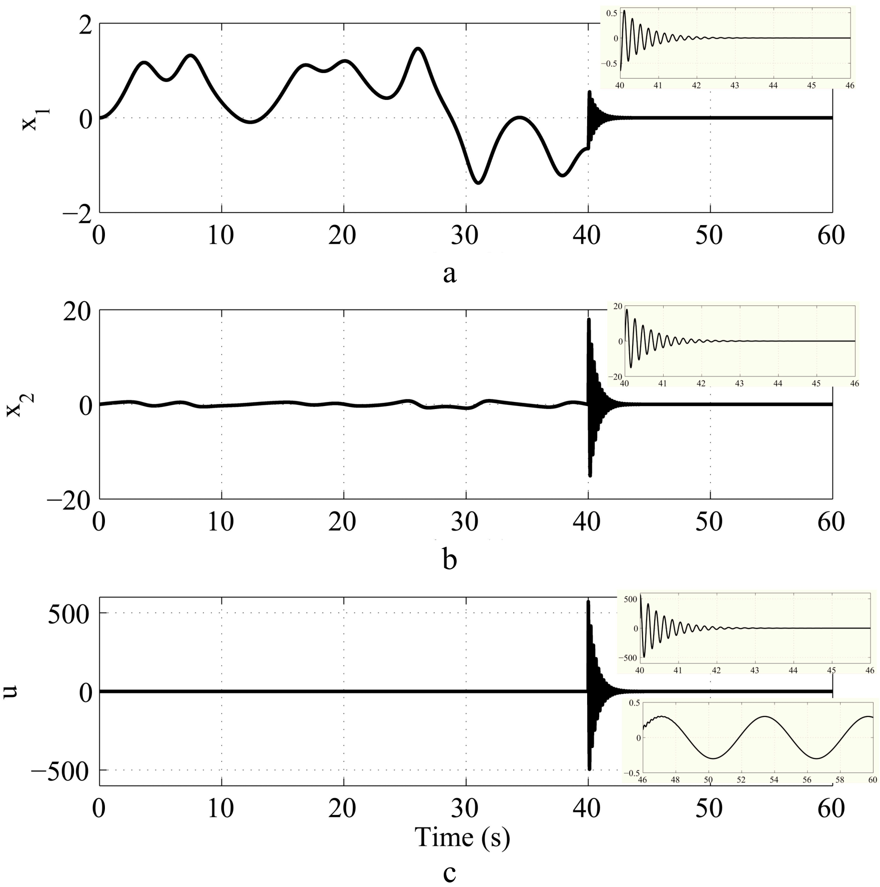


Fig. 6. *Responses of x_1, x_2, and u in Example 3*

a Response of x_1

b Response of x_2

c Response of u

The response of the ratio $\sqrt{\int_0^t z^T zdt} / \sqrt{\int_0^t w^T wdt}$ under zero initial conditions and $w = 500e^{-1.6t}\sin(2.5t)$ is shown in Fig. 7. The ratio is less than 0.1030, which is smaller than $\gamma = 1.2$.

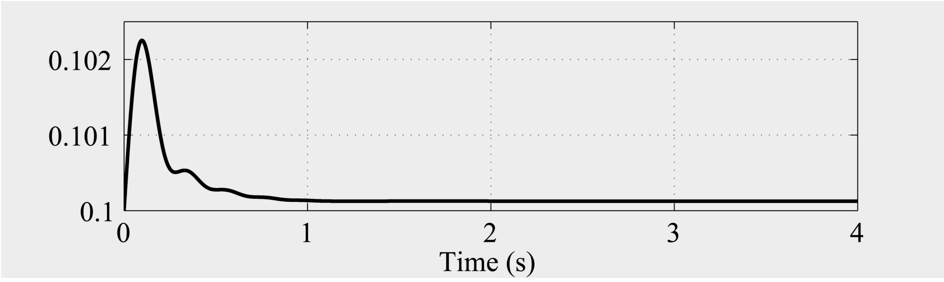


***Fig. 7.** Response of the ratio* $\sqrt{\int_0^t z^T zdt} / \sqrt{\int_0^t w^T wdt}$ *in Example 3*

5. Conclusions

In this paper, an effective H_∞ fuzzy PID controller design method has been proposed and studied. First, the fuzzy PID controller design problem is transformed into that of the fuzzy SOF controller design. Next, the proposed ILMI algorithm is used to design the fuzzy SOF controller. Finally, the fuzzy PID controller can be recovered from the fuzzy SOF controller. Simulation examples have been given to demonstrate the effectiveness of our approach.

6. Acknowledgments

The authors would like to thank the associate editor and anonymous reviewers for their constructive comments, which have significantly improved the paper. This work was supported by National Natural Science Foundation of China (No. 61503096) Heilongjiang Postdoctoral Fund (No. LBH-Z14101) and the Key Laboratory Program (No. BAQQ5780001515 and BAQQ5780005415).

7. References

[1] Åström K. J., Hägglund T.: 'PID Controllers: theory, design and tuning' (Research Triangle Park, Instrument Society of America, 1995)

[2] J. G. Ziegler, N. B. Nichols: 'Optimum settings for automatic controllers', Transaction of ASME, 1942, **115**, 2B, pp 759-768

[3] Zhuang M., Atherton D. P.: 'Automatic tuning of optimum PID controllers', Proceedings of IEE, Part D, 1993, **140**, 3, pp 216-224

[4] Chen G.: 'Conventional and fuzzy PID controllers: An overview', Intelligent control and system, 1996, **1**, 2, pp 235-246

[5] Ying H., Siler W., Buckley J. J.: 'Fuzzy control theory: a nonlinear case', Automatica, 1990, **26**, 3, pp 513-520

[6] Ying H.: 'A fuzzy controller with linear control rules is the sum of a global two-dimensional multilevel relay and a local nonlinear proportional-integral controller', Automatica, 1993, **29**, 2, pp 499–505

[7] Ying H.: 'The simplest fuzzy controllers using different inference methods are different nonlinear proportional-integral controllers with variable gains', Automatica, 1993, **29**, 6, pp 1579-1589

[8] Carvajal J., Chen G., Ogmen H.: ‘Fuzzy PID controller: Design, performance evaluation, and stability analysis’, Information Sciences, 2000, **123**, 3-4, pp 249-270
[9] Guerra R. E. H., Braess G. S., *et al.*: ‘Using circle criteria for verifying asymptotic stability in PI-like fuzzy control systems: application to the milling process’, IEE Proceedings Control Theory and Applications, 2003, **150**, 6, pp 619-627
[10] Misir D., Malki H. A., Chen G.: ‘Design and analysis of a fuzzy proportional-integral-derivative controller’, Fuzzy Sets and systems. 1996, **79**, 3, pp 297-314
[11] Mudi R. K., Pal N. R.: ‘A robust self-tuning scheme for PI- and PD-type fuzzy controllers’, IEEE Transactions on Fuzzy Systems, 1999, **7**, 1, pp 2-16
[12] Li T. H. S., Shieh M. Y.: ‘Design of a GA-based fuzzy PID controller for non-minimum phase systems’, Fuzzy Sets and Systems, 2000, **111**, 2, pp 183-197
[13] Li C. Y., Jing W. X.: ‘Fuzzy PID controller for 2D differential geometric guidance and control problem’, IET Control Theory and Applications, 2007, **1**, 3, pp 564-571
[14] Kim J. H., Oh S. J.: ‘A fuzzy PID controller for nonlinear and uncertain systems’, Soft Computing, 2000, **4**, 2, pp 123-129
[15] Xu J. X., Hang C. C., Liu C.: ‘Parallel structure and tuning of a fuzzy PID controller’, Automatica, 2000, **36**, 5, pp 673-684
[16] Ying H.: ‘Theory and application of a novel fuzzy PID controller using a simplified Takagi-Sugeno rule scheme’, Information Sciences, 2000, **123**, 3, pp 281-293
[17] Ding Y. S., Ying H., Shao S. H.: ‘Typical Takagi-Sugeno PI and PD fuzzy controllers: analytical structures and stability analysis’, Information Sciences, 2003, **151**, 3, pp 245-262
[18] Sio K. C., Lee C. K.: ‘Stability of fuzzy PID controllers’, IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans. 1998, **28**, 4, pp 490-495
[19] Aracil J., Gordillo F.: ‘Describing function method for stability analysis of PD and PI fuzzy controllers’, Fuzzy Sets and Systems, 2004, **143**, 2, pp 233-249
[20] Ban X. J., Gao X. Z., Huang X. L., *et al.*: ‘Stability analysis of the simplest Takagi-Sugeno fuzzy control system using circle criterion’, Information Sciences, 2007, **177**, 20, pp 4387-4409
[21] Cao K. R., Gao X. Z., Huang X. L., *et al.*: ‘Stability analysis of a type of Takagi-Sugeno PI fuzzy control systems using circle criterion’, International Journal of Computational Intelligence Systems, 2011, **4**, 2, pp 196-207.
[22] Precup R. E., Preitl S., Korondi P.: ‘Fuzzy controllers with maximum sensitivity for servosystems’, IEEE Transactions on industrial electronics, 2007, **54**, 3, pp 1298-1310
[23] M. Petrov, I. Ganchev, A. Taneva. ‘Fuzzy PID control of nonlinear plants’. Proceedings of 2002 First International IEEE Symposium "Intelligent Systems", Varna, Bulgaria, September 2002, pp: 30-35
[24] Zheng F., Wang Q. G., Lee T. H., *et al*: ‘Robust PI controller design for nonlinear systems via fuzzy modeling approach’, IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans, 2001, **31**, 6, pp 666-675
[25] Zheng F., Wang Q. G., Lee T. H.: ‘On the design of multivariable PID controllers via LMI approach’, Automatica, 2002, **38**, 3, pp 517-526
[26] He Y., Wang Q. G.: ‘An improved ILMI method for static output feedback control with application to multivariable PID control’, IEEE Transactions on Automatic Control, 2006, **51**, 10, pp 1678-1683
[27] Zhou K., Doyle J. C., Glover K.: ' Robust and Optimal Control’ (Prentice-Hall, New York, 1995)
[28] Lo J. C., Lin M. L.: ‘Robust H_∞ nonlinear control via fuzzy static output feedback’, IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, 2003, **50**, 11, pp 1494-1502
[29] Tanaka K., Wang H. O.: ‘Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach’ (Wiley, New York, 2001)
[30] Chen C.T: ‘Linear System Theory and Design’ (Rinehart and Winston, Holt, 1999)
[31] Cao Y. Y., Lam J., Sun Y. X.: ‘Static Output Feedback Stabilization: An ILMI Approach’, Automatica, 1998, **34**, 12, pp 1641-1645
[32] Wu H. N., Zhang H. Y.: ‘Reliable mixed H_∞ fuzzy static output feedback control for nonlinear systems with sensor faults’, Automatic, 2005, **41**, 11, pp 1925-1932

[33] Dong J. X., Yang G. H.: 'Static output feedback H_∞ control of a class of nonlinear discrete-time systems', Fuzzy Sets and systems, 2009, **160**, 19, pp 2844-2859
[34] Qiu J. B., Feng G., Gao H. J.: 'Static-output-feedback H_∞ control of continuous-time T-S fuzzy affine systems via piecewise Lyapunov functions', IEEE Transactions on Fuzzy Systems, 2013, **21**, 2, pp 245-261
[35] Chang X. H., Zhang L., Park J. H.: 'Robust static output feedback control for uncertain fuzzy systems', Fuzzy Sets and Systems, 2015, **273**, 15, pp 87-104
[36] Tuan H. D., Apkarian P., Narikiyo T.,: 'Parameterized linear matrix inequality techniques in fuzzy control system design', IEEE Transactions on Fuzzy Systems, 2001, **9**, 2, pp 324-332
[37] Boyd S., Ghaoui L. E., Feron E., *et al*: 'Linear Matrix Inequalities in System and Control Theory' (PA: SIAM, Philadelphia, 1994)
[38] Cao S. G., Rees N. W., Feng G.: 'Stability analysis and design for a class of continuous-time fuzzy control systems', International Journal of Control, 1996, **64**, 6, pp 1069-1087
[39] Cao S. G., Rees N. W., Feng G.: ' H_∞ control of nonlinear continuous-time systems based on dynamical fuzzy models', International Journal of Systems Science, 1996, **27**, 9, pp 821-830